Problem 1. A card is drawn at random from a deck consisting of cards numbered 2 through 10. A player wins 1 dollar if the number on the card is odd and loses 1 dollar if the number is even. What is the expected value of his winnings?

Problem 2. A card is drawn at random from a deck of playing cards. If it is red, the player wins 1 dollar; if it is black, the player loses 2 dollars. Find the expected value of the game.

Problem 3. In a class there are 20 students: 3 are 5' 6", 5 are 5'8", 4 are 5'10", 4 are 6', and 4 are 6' 2". A student is chosen at random. What is the students expected height?

Problem 4. In Las Vegas the roulette wheel has a 0 and a 00 and then the numbers 1 to 36 marked on equal slots; the wheel is spun and a ball stops randomly in one slot. When a player bets 1 dollar on a number, he receives 36 dollars if the ball stops on this number, for a net gain of 35 dollars; otherwise, he loses his dollar bet. Find the expected value for his winnings.

Problem 5. In a second version of roulette in Las Vegas, a player bets on red or black. Half of the numbers from 1 to 36 are red, and half are black. If a player bets a dollar on black, and if the ball stops on a black number, he gets his dollar back and another dollar. If the ball stops on a red number or on 0 or 00 he loses his dollar. Find the expected winnings for this bet.

Problem 6. A die is rolled twice. Let \( X \) denote the sum of the two numbers that turn up, and \( Y \) the difference of the numbers (specifically, the number on the first roll minus the number on the second). Show that \( E(XY) = E(X)E(Y) \). Are \( X \) and \( Y \) independent?

Problem 7. A royal family has children until it has a boy or until it has three children, whichever comes first. Assume that each child is a boy with probability 1/2. Find the expected number of boys in this royal family and the expected number of girls.

Problem 8. Choose a number \( U \) from the unit interval \([0, 1]\) with uniform distribution. Find the cumulative distribution and density for the random variables.
   (a) \( Y = U + 2 \).
   (b) \( Y = U^3 \).

Problem 9. Choose a number \( U \) from the interval \([0, 1]\) with uniform distribution. Find the cumulative distribution and density for the random variables
   (a) \( Y = 1/(U + 1) \).
   (b) \( Y = \log(U + 1) \).

Problem 10. Let \( U, V \) be random numbers chosen independently from the interval \([0,1]\). Find the cumulative distribution and density for the random variables
   (a) \( Y = \max(U, V) \).
   (b) \( Y = \min(U, V) \).

Problem 11. A number \( U \) is chosen at random in the interval \([0, 1]\). Find the probability that
   (a) \( R = U^2 < 1/4 \).
   (b) \( S = U(1 - U) < 1/4 \).
   (c) \( T = U/(1 - U) < 1/4 \).

Problem 12. A point \( P \) in the unit square has coordinates \( X \) and \( Y \) chosen at random in the interval \([0, 1]\). Let \( D \) be the distance from \( P \) to the nearest edge of the square, and \( E \) the distance to the nearest corner. What is the probability that
   (a) \( D < 1/4 \).
   (b) \( E < 1/4 \).
Problem 13. Let $X$ be a random variable with cumulative distribution function $F$ strictly increasing on the range of $X$. Let $Y = F(X)$. Show that $Y$ is uniformly distributed in the interval $[0, 1]$.

Problem 14. Assume that, during each second, a Dartmouth switchboard receives one call with probability .01 and no calls with probability .99. Use the Poisson approximation to estimate the probability that the operator will miss at most one call if she takes a 5-minute coffee break.

Problem 15. A baker blends 600 raisins and 400 chocolate chips into a dough mix and, from this, makes 500 cookies. (a) Find the probability that a randomly picked cookie will have no raisins. (b) Find the probability that a randomly picked cookie will have exactly two chocolate chips. (c) Find the probability that a randomly chosen cookie will have at least two bits (raisins or chips) in it.

Problem 16. An advertiser drops 10,000 leaflets on a city which has 2000 blocks. Assume that each leaflet has an equal chance of landing on each block. What is the probability that a particular block will receive no leaflets?

Problem 17. A manufactured lot of buggy whips has 20 items, of which 5 are defective. A random sample of 5 items is chosen to be inspected. Find the probability that the sample contains exactly one defective item (a) if the sampling is done with replacement. (b) if the sampling is done without replacement.

Problem 18. You are presented with four different dice. The first one has two sides marked 0 and four sides marked 4. The second one has a 3 on every side. The third one has a 2 on four sides and a 6 on two sides, and the fourth one has a 1 on three sides and a 5 on three sides. You allow your friend to pick any of the four dice he wishes. Then you pick one of the remaining three and you each roll your die. The person with the largest number showing wins a dollar. Show that you can choose your die so that you have probability 2/3 of winning no matter which die your friend picks.

Problem 19. Prove the following binomial identity

$$\binom{2n}{n} = \sum_{j=0}^{n} \left( \binom{n}{j} \right)^2$$

Hint: Consider an urn with $n$ red balls and $n$ blue balls inside. Show that each side of the equation equals the number of ways to choose $n$ balls from the urn.

The End.